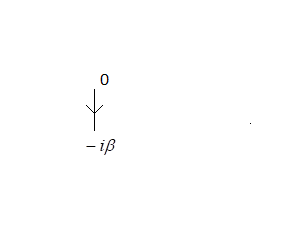
**Perturbative Expansion of Green’s Function**

So the main quantity of interest vis a vis thermally averaged GF, and even the other ones, is the retarded GF because this is the one which shows up in transport properties, as we’ll see in NESM. So one possibility is to develop an expansion based off of the differential equation satisfied by the GF. This would be done the same as how we did it in the regular QM case. But I’ll skip that and just go into the Wick approach.

**Thermal averaged complex time GC\***

The expansion of the real time GF’s is complicated. Matsubara recognized there is an easier way. Let’s consider again, our complex time GF. Matsubara saw that instead of doing an expansion involving a time-ordered contour going along the real and imaginary axes, it’d be a lot easier to do an expansion of a complex time-ordered GF, along the imaginary time axis alone.



And the important thing is that it would turn out to be a completely self-consistent expansion, involving only the (imaginary/complex) time-ordered GF. Just to see, let’s consider, emphasizing that t = -iτ in the notation below:



Let’s suppose τ1 > τ2. Then we have:



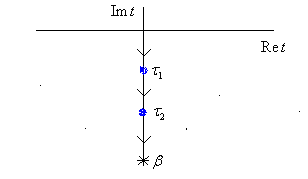
and if τ2 > τ1, we have:



So either way, we have:



and the time-ordering contour looks like,



And so we can straightaway calculate this quantity. It just involves a simple time-ordering, along the imaginary axis t = -iτ, from τ = 0 to τ = β. And so we can use Wick’s theorem, perturbatively pull down the terms in the S exponent, and binarily group time-ordered expectations. And the denominator should cancel out the vacuum bubbles.

And yeah have to divide top/bottom by Tre-βK0, technically, to do the Wick expansion in terms of non-interacting GF’s.

Lastly, we’ll note the retarded GF can be obtained from the complex time GF by simple analytic continuation (with the expressed caveats in that file) iωn → ω + i0+ (or iνn → ω + i0+). This makes this approach much easier in general than the real time approach. Still, for non-equilibrium thermodynamics, I think the real-time approach is often utilized.

**Feynman Rules for Lattice**

So let’s consider a Hamiltonian like this:



and discuss the expansion of the GF:



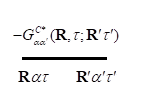
where S is:



and the x’s are now interaction picture guys.

**Real Space Rules**

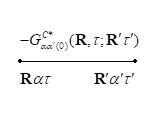
So we have just the one GF now:



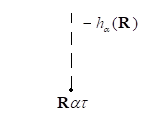
and the two external points:



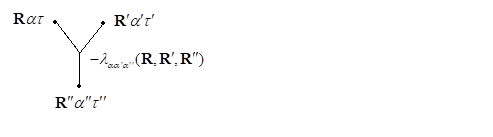
which we connect with the bare GF,



and the one external potential vertex:



and the interaction vertex, cubic in this example, would be:



So the general procedure is as usual:

**Topology**

Connect all topologically distinct diagrams together, associating with each element in the diagram the indicated term. Leave out vacuum bubbles, i.e., diagrams that aren’t connected to either t or t´.

**Equal time issues**

There are no equal time issues to be concerned about to my knowledge.

**Signs/Numerical Factors**

The topologically distinct diagrams show up in many different equivalent ways in the direct S-matrix expansion. And so we have to multiply each by the number of times it would appear in the expansion, divided by the 1/n! coming from the S-matrix expansion. The rules for calculating this number are, I think as follows. And it is probably necessary to stipulate the whatever λmm´m´´(R,R´,R´´) is, it is symmetric w/r to interachange of any two coordinate+index pairs, say for instance (m,R) with (m´´,R´´). This probably makes λmm´m´´(R,R´,R´´) a scalar.

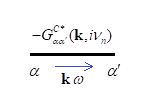


**Sum**

Then sum/integrate over all internal positions/times/indices. There is a simplification to be had if we have a homogeneous isotropic medium. Then GF0 and GF are diagonal in their indices αα´. In that case, if we were to be interested in calculating G11, say, then the δαα´ attached to the GF would, along with the requisite index sum, end up fixing all GF’s in the diagram to that index G(0)11. So then we wouldn’t have to worry about any index sum per se’. Except in the case that we have closed loops. Then it seems to me to be similar to the fermion loop discussed in the identical particles file, and we’d have to multiply such loops by a factor of 3 (presuming GF11 = GF22 = GF33 which seems to be necessary in such case).

**Fourier Space Rules**

So we have the GF:



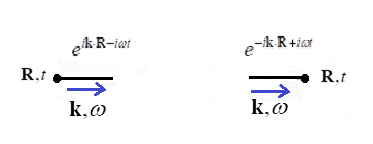
where these G’s are the Fourier transforms of the real space guys,



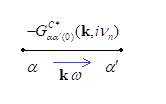
and the external points:



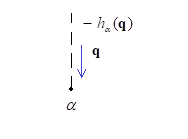
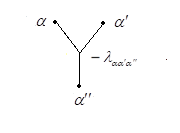
These carry the Fourier transform variable. And if we’re ultimately interested in the real space, time GF, then we must add the following factors to each:



and we connect them with the bare GF:



and single particle potential, and interaction vertex:

and we’d impose momentum (wavenumber), energy (frequency) conservation at each vertex. Can see the Fourier transform file for some justification of these rules.

**Topology**

Same as above.

**Equal time issues**

No particular issues here again.

**Signs**

Same old Feynman rules as above.

**Sum**

Then sum/integrate over all independent internal momenta wavenumbers/energies frequencies.



including the external one if we’re interested in the real space GF.

**Feynman Rules for Interacting Indistinct Particles**

Let’s say we have:



where,



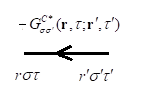
And consider the (complex) time ordered green’s function:



Then,

**Real Space Rules**

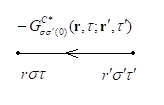
So we have just one GF now



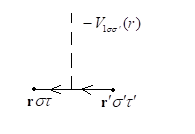
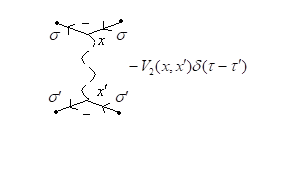
and one set of external points:



which we connect via bare GF:



and vertices:

Then the general procedure is as follows…

**Topology**

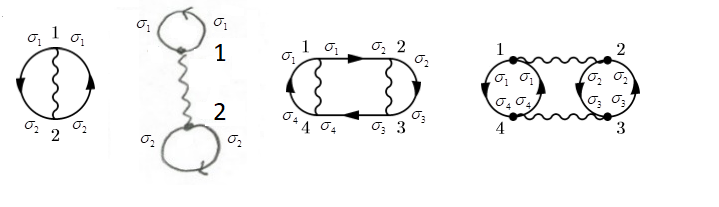
Connect all topologically distinct, connected (meaning all end points must be connected to each other though not necessarily to the rest of the diagram – meaning vacuum bubbles allowed at this stage), diagrams together, associating each element in diagram with the indicated term. Note the order of indices is important, Gσσ´(xk,xj) ≠ Gσ´σ(xj,xk), so always order the positions against the directions of the arrows.

**Equal time issues**

There are two equal time issues, depicted below. As usual, should interpret Gσσ´(x,τ;x´,τ) as Gσσ´(x,τ;x´,τ+), and in particular Gσσ(x,τ;x,τ) as Gσσ(x,τ;x,τ+) = -<ψσ(x,τ)ψ†σ(x,τ+) > = ε<ψ†σ(x,τ)ψσ(x,τ)> = ε<nσ(x)> which is the thermal average of the density, times ε. Note that the ψ’s would come naturally in the form T{ψ†(s)ψ(s)}, say. But then we’d have to permute them to (-ε)T{ψ(s)ψ†(s)}, so that we can turn it into a GF. But then we’d interpret this as (-ε)TC{ψ(s)ψ†(s+)}, which would change it back to (-ε)(-ε)ψ†(s)ψ(s). = ψ†(s)ψ(s). All this is to say that to interpret the ψ combination as a GF we have to already permute them out of order, which introduces a (-ε). So the (-ε) we introduce by interpreting the latter time as s+ just cancels out the first (-ε) with another one.

**Signs/Numerical Factors**

Each diagram is associated with a numerical factor, because each topologically distinct diagram comes from a variety of contractions and because we must factor in the 1/n! term coming from the S-matrix expansion. Basically, diagrams with external legs seem to have a simple net factor of unity. Bubbles have a symmetry factor of 1/p, where p is the number of permutations (including the identity permutation) of points that result in the same diagram [note discussion below presumes all vertices are *same* sign]. For instance,



We can do the permutation (12) and so the first guy gets a factor of ½. Same with the second guy. The third gets a factor of ½ as well because all we can do is the permutation (13)(24). Can verify that flipping diagram about horizontal and vertical would result in this permutation and so it works. Not sure why (14)(23) wouldn’t also be acceptable, but for the possible fact that there doesn’t seem to be a corresponding set of reflections that would result in this permutation, like there was for the previous. Well, maybe because in present state, we have propagator going from 1 → 4, and 3 → 2, and if you switch the numbers, then the propagators would be going 4 → 1 and 2 → 3. So that’s different. Then the last guy has the permutations, (12)(34), (13)(24), and (14)(23). So it gets a factor of ¼.

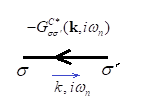
(Fermion) loops have there own special sign factors. Each fermion loop get a factor of (-ε), which would be just 1 for a boson loop – this has to do with intracacies of transposing the operators to put them in proper order. The four diagrams have 1, 2, 1, 2 Fermion loops respectively.

**Sum**

Then integrate/sum over all internal vertices, including spin ones. Must sum over spin indices if fermion too. Apropos the spin sum, if spin is conserved by H, and presuming G0σσ´ to be diagonal in spin index as ours is, then we’ll note that if we start off trying to calculate G↑↑, then the δσσ´ attached to our GF, coupled with our spin sum, will simply have the effect of enforcing all other GF’s in the diagram to also be G0↑↑. Likewise if we tried to calculate the spin down GF. So basically we can ignore the spin sum part. Except in one instance: fermion loops. In these parts of the diagram, the GF’s are in some sense disconnected from the external points, and since they circle back on themselves we end up with a term like Σσ1,σ2,σ3G(0)σ1σ2G(0)σ2σ3G(0)σ3σ1. And this reduces to Σσ3G(0)σ3σ3G(0)σ3σ3G(0)σ3σ3 = G(0)↑↑G(0)↑↑G(0)↑↑ + G(0)↓↓G(0)↓↓G(0)↓↓ = 2[G(0)↑↑G(0)↑↑G(0)↑↑], say. And so we just need to multiply our diagrams by 2 (or 2s+1 if higher spin) to account for these loops. The simple factor of two *does* presume that G(0)↑↑ = G(0)↓↓, however. This could feasibly not be so if we included a Bz in our unperturbed H0. So if not then we’d just replace the trace by whatever that works out to be.

**Fourier Space Rules**

So we have our single GF again:



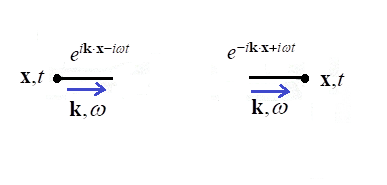
where these G’s are the Fourier transforms of the real space guys,



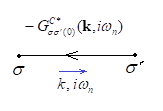
and external points:



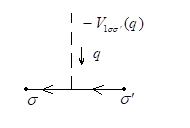
These carry the Fourier transform variable. And if we’re ultimately interested in the real space, time GF, then we must add the following factors to each:



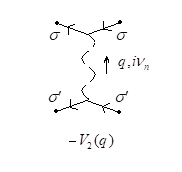
and we connect them with the bare GF:



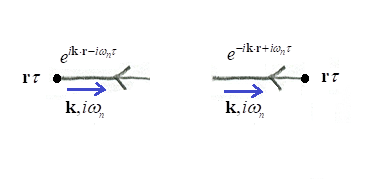
And we associate single particle potential with:



There is no ω-energy flowing down the line because no time dependence. And the two particle potential with (assuming translationally invariant and no time-dependence):



We still have iνn flowing down the line because of the artificial time-dependence brought in by the δ(t-t´) we put on each interaction potential. Note it’s a bosonic frequency, though, as energy conservation demands, because we cannot have electron lines with fermionic frequencies if we have a fermionic interaction frequency traveling into it (EM interaction is carried by bosons too – is this a coincidence?). And we also have external legs if want the real space GF out of this:



**Topology**

Connect all topologically distinct, fully connected (meaning no vacuum bubbles), diagrams together, associating with each element in the diagram the indicated term. Remember that all energy-momentum labels **k**, ω must be going in the same way w/r to the GF arrow. They must all be going against the arrows (or with them I suppose). And then we conserve energy/momentum at each vertex.

**Equal time issues**

Apropos the equal time issue, in Fourier space this would be handled as:



So we need to include a factor of exp(iωn0+) for each bare Green’s function closed in on itself. Fundamentally, the problem occurs when the temporal argument of the bare Green’s function is zero.

**Signs/Numerical Factors**

Same as above.

**Sum**

Then sum/integrate over all independent momenta - wavenumbers/energies-frequencies/indices. If finite spatial transform, then the guy on the left. If continuous spatial transform, then the guy on the right:



and include the external frequency/wavenumber if want the real space GF.

**Feynman Rules for Photons**

I’ll just say it’s basically the same as the rules for phonons.

**Justification of some of the assertions alluded to above**

Inserting our expression for S into the Green’s function formula, we find



where < > means Tre-βK0{ }. All operators’ time development is implicitly taken to be in the interaction picture, and A, B are presumed also to be in said picture. Recalling the discussion in the 2nd quantization file, especially Wick’s theorem, we see that to evaluate the expectation in the numerator, we can simply decompose it into the sum of all permutations of full contractions (not forgetting to append the appropriate minus sign if necessary to bring the contracted items together – which is what the fermion loop factor is supposed to address). Note that this result applies to any perturbation – not just interactions.

**The -1 associated with each potential**

So we can see above that -1 and V each occur to the same power. So we can slide the -1 next to the potential, and this is where the -1 factor comes from.

**The δ(τ-τ´) associated with the V2 interaction (indistinguishable particles)**

Let’s write out the term in more detail,



So we have that we can simplify our rules and associate each end of the interaction vertex with a separate space-time point, if we include the δ function. Of course we’ve also doubled the number of time integrations as well, but the simplicity is worth it I imagine. So now we can write, to the extent that we’re dealing with V2 interactions:

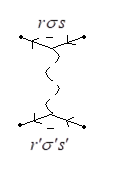


where,



**Elimination of the (1/2) associated with the V2 interaction (indistinguishable particles)**

These goes away because there are two equivalent ways to connect to the interaction. We can have one external leg going to rσs and the other to r´σ´s´, or vice versa.



But these are both the same diagram because the interaction is an even function of rσs – r´σ´s´. So we can just connect one way and multiply by two. This is for case when have external legs connected to a diagram.

**That A and B must be topologically connected to each other (indistinguishable particles)**

Going to use a bit of notation. First, I’m going to go back to treating each operator as just a single function of time b/c that’s more compact here. And then, since each operator is associated with an integration over its time argument (coming from it being evaluated in the integration picture), I’ll write:



where the operator thing on top is meant to remind us we have to integrate over time. Okay, now note that if A and B are single particle creation/annihilation operators (in no particular order), and because V1 and V2 contain even powers of creation/annihilation operators, there is no way to connect A to one set and B to another, because that would mean A + set 1 would have an odd number of what A is, but an even number of what B is. And B + set 2 would have an odd number of whatever B is, but an even number of whatever A is. And any expectation with a mismatch of powers of ψ and ψ† will be zero. So A has to be connected topologically to B. So we have:



But note that if A and B were say bilinear operators, like we have for phonons and photons, then the two wouldn’t have to be connected. And even regardless, this doesn’t mean that A and B can’t be connected to each other, but disconnected from the rest of the diagram – like vacuum bubbles. This is what the next part is about.

**Elimination of vacuum bubbles**

So we have:



Now we will apply Wick’s theorem to break the many particle expectation into sums of all possible contractions. Looking at the nth order term, we can classify the possible contractions into n distinct sets according to the number m = 0,1,…,n of V’s that are contracted connectedly (because they have to be as argued above) with the external points t, tʹ and (n-m) V’s which are contracted (in no particular way) amongst themselves. Note that the multiplicity of the mth set is equal to:



equal to the number of ways to choose m V’s to fully contract connectedly to the external points, and (n – m) V’s to contract amongst themselves (can have bubbles among the last set). All these permutations within the mth set are all equivalent because we integrate over all the s’s which are just dummy variables. So we have:



(note again, the multiplicity just refers to the number of ways we could get the left < > with the external points, not the number of ways the right < > can be broken up into equivalent terms) or in other words,



We now plug this back into the formula for the Green’s function



Now we want to somehow separate the connected diagrams from bubbles. To this end we’ll reorganize the sum as follows…



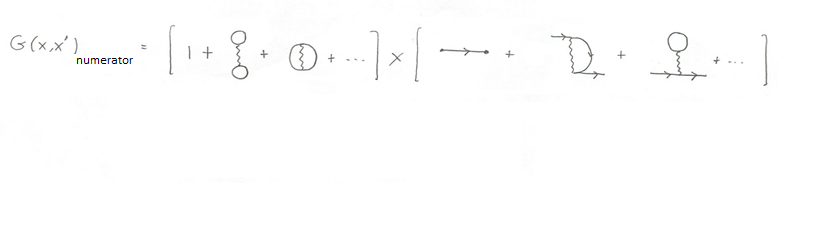
So we have completely factorized the connected and disconnected diagrams, and we can write this as, summing the product



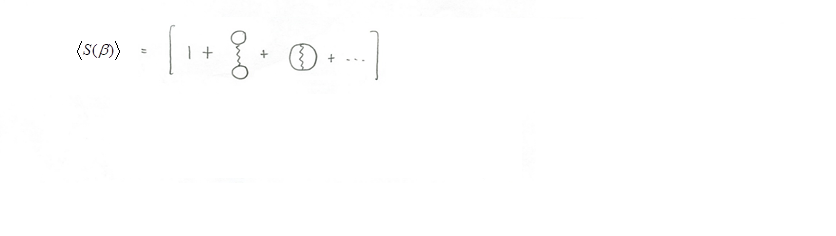
We’ll recognize the right < > as just the denominator of the GF. And so we have:



In pictures, basically we have (illustrated for two point GF and V2 interaction):



and



Thus the sum of all vacuum bubbles will cancel in the numerator and denominator and we’ll be left simply with the sum of all fully connected diagrams – as we know. So we will see that we can incorportate a self energy into the expression for the Green’s function, as we have been able to do before.

**Elimination of topologically equivalent diagrams, basically the 1/n! factor (indistinguishable particles)**

The next simplification occurs when we realize that



because in the second line there will be n! topologically indistinct diagrams (meaning that the difference in the diagrams consists merely in a relabelization of the internal points). The reason is the following. Suppose that we have a certain full contraction. Then imagine that we leave the contraction lines ‘fixed’ and just permute the order of the V’s, in the same fashion that we do here



(the contraction line stays fixed between the first and third items, we just permute the arrangement of the terms). Then this new contraction which presently would be included as a distinct term in the perturbative expansion will just give us the same value as the unpermuted one because we have effectively only changed the labels on the s’s, and they are dummy variables. Now there are n! ways to permute the positions of the H’s, so the n! topologically indistinct diagrams all give the same value. But note this happens only for diagrams that are fully connected. This doesn’t apply to vacuum bubbles, which instead retain a symmetry factor of 1/p, because our argument here will have over estimated the number of different ways we can construct the bubble. But of course, we have no vacuum bubbles in this GF by the prior argument.

**What about the sign convention?**

We have yet to prove that we get (-1) for each fermion loop. But oh well.